Surprises in the AdS algebraic curve constructions — Wilson loops and correlation functions

Romuald A. Janik ^{a,b,*}, Paweł Laskoś-Grabowski ^{a,c,†}

^{a} Institute of Physics, Jagiellonian University
ul. Reymonta 4, 30-059 Kraków, Poland

^{b} Institute for Advanced Studies
The Hebrew University of Jerusalem
Givat Ram Campus, 91904 Jerusalem, Israel

^{c} Institute for Theoretical Physics, University of Wrocław
pl. Maxa Borna 9, 50-204 Wrocław, Poland

Abstract

The algebraic curve (finite-gap) classification of rotating string solutions was very important in the development of integrability through comparison with analogous structures at weak coupling. The classification was based on the analysis of monodromy around the closed string cylinder. In this paper we show that certain classical Wilson loop minimal surfaces corresponding to the null cusp and $q\bar{q}$ potential with *trivial monodromy* can, nevertheless, be described by appropriate algebraic curves. We also show how a correlation function of a circular Wilson loop with a local operator fits into this framework. The latter solution has identical monodromy to the pointlike BMN string and yet is significantly different.

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*e-mail: `romuald@th.if.uj.edu.pl`
†e-mail: `plg@th.if.uj.edu.pl`

1 Introduction

The AdS/CFT correspondence, which postulates the equivalence of $\mathcal{N} = 4$ Super-Yang-Mills theory and superstrings in $AdS_5 \times S^5$ spacetime, provides a unique opportunity for solving, for the first time, an interacting four-dimensional gauge theory (see the recent review [1]).

Currently we have a very good understanding of the spectral problem, i.e. of anomalous dimensions of local gauge theory operators at any coupling, which gets translated to the energy levels of the closed string in $AdS_5 \times S^5$ spacetime, i.e. the energy levels of the corresponding worldsheet quantum field theory. The answer is formulated in terms of Thermodynamic Bethe Ansatz [2, 3, 4] or NLIE [5, 6] equations for that integrable worldsheet QFT. Of course, there are still many issues which have not been worked out, like the structure of source terms in these equations for arbitrary operators, but still our understanding is much more complete than for other observables.

A significant step in the above progress was the development of the algebraic curve (finite-gap) classification of classical spinning string solutions [7, 8, 9, 10] and a comparison of the emerging structures with similar classical analysis of the Bethe equations coming from a gauge theory spin chain description at weak coupling.

Because of this theoretical importance, our motivation was to investigate whether one could employ similar algebraic curve methods for other classes of classical string solutions in $AdS_5 \times S^5$ which also have an important gauge-theoretical meaning – Wilson loops and correlation functions.

Since the method of [7] was based on a thorough analysis of the analytical properties of monodromy around a noncontractible loop going around the closed string cylinder, it would seem that there is no chance of applying similar constructions to Wilson loop minimal surfaces on which all loops are contractible and hence have trivial monodromy. Indeed, the use of integrability for polygonal null Wilson loops related to scattering amplitudes [11, 12, 13] relied on completely different methods specific to that particular setup.

In this paper we will show that, nevertheless, one can associate algebraic curves to such classical solutions as the null cusp minimal surface or the $q\bar{q}$ potential Wilson loop and conversely, one can reconstruct the full target space solution purely algebraically from the given algebraic curve.

The consideration of classical solutions corresponding to correlation functions of local operators [14, 15, 16, 17] (and possibly also other objects like Wilson loops [18]) poses a different kind of question to the classical algebraic curve construction of [7]. For these solutions, the monodromy around a given puncture should, *by definition*, be identical to the monodromy (pseudomo-

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mentum) of the spinning string corresponding to the local operator at the puncture. Therefore, the starting point of the construction of [7] would be identical for the ordinary spinning string and for the correlation function. Yet the classical solutions are significantly different. This shows that there should be an enormous freedom in the construction of solutions with pre-scribed pseudomomentum, going far beyond the folklore that such solutions are parametrized essentially by a finite-dimensional Jacobian of the relevant algebraic curve. The understanding of how this freedom may arise was one of the motivations for this paper. We investigate here the correlation function of a circular Wilson loop with the $\text{tr}\, Z^J$ local operator and show how the algebraic curve description differs from the pointlike string corresponding to $\text{tr}\, Z^J$.

Finally, another motivation for describing Wilson loops in the same set-ting as closed strings was to understand from this perspective possible links between the two quite different kinds of solutions. An outstanding example of such a relationship is the link between the large spin limit of the GKP folded string [19] and the null cusp Wilson loop [20]. This relationship al-lowed for the identification of the cusp anomalous dimension appearing in the Wilson loop with the large spin asymptotics of the anomalous dimen-sions of twist-two operators. Hence one could use the well developed Bethe ansatz methods (in this limit wrapping corrections do not contribute) at any coupling (for the spinning string) to gain all-order information on the Wilson loop.

Having a unified description of both kinds of solutions may help un-derstanding such relationships and perhaps uncover new ones. One of the motivations for the present work was to investigate the possibility of such a relationship with the $q\bar{q}$ Wilson loop minimal surface. The simplest de-formation yields unfortunately just the generalized $q\bar{q}$ Wilson loop of [21], yet perhaps there might be a more intricate generalization through e.g. a degeneration of a genus-2 curve.

Let us note, that as this paper was being prepared, significant progress was made in the exact evaluation of the $q\bar{q}$ potential [22, 23]. It would be very interesting to understand whether the algebraic curve for the $q\bar{q}$ potential identified here has any interpretation in these approaches.

The plan of the paper is as follows. In section 2 we review briefly the classical integrability of the AdS_3 σ-model, and in section 3, the algebraic curve (finite-gap) description of spinning strings and the reconstruction pro-cedure which enables one to obtain the target-space solution from the al-gebraic curve. In section 4 we summarize the key questions of the present paper. In section 5 we present our example classical solutions and identify the corresponding algebraic curves, while in the following section, we show

that one can indeed reconstruct the original Wilson loop minimal surface or correlation function solution just from the knowledge of the algebraic curves and some minimal structural assumptions. We close the paper with conclusions and some remarks on possible applications. In the appendices, we have collected some formulas for elliptic functions and we identify the classical solutions arising from algebraic curves approximating the null cusp (here we obtain the GKP folded string) and the $q\bar{q}$ potential (in this case we obtain the known generalized minimal surface in global AdS_3).

2   The AdS_3 σ-model and its integrability

It is very well known that the full $AdS_5 \times S^5$ σ-model is integrable [24]. In this paper, for simplicity, we will concentrate on its subsector, the AdS_3 σ-model, which is also classically integrable by itself. In order to exhibit integrability, it is most convenient to rewrite its action in terms of group elements:

$$S_{AdS_3} = \frac{\sqrt{\lambda}}{4\pi} \int \operatorname{tr} j\bar{j} \, d^2w \tag{1}$$

where w and $\bar{w}$ are the worldsheet coordinates,[1] the currents are given by

$$j = g^{-1}\partial g \qquad \bar{j} = g^{-1}\bar{\partial} g \tag{2}$$

with the group element having one of the following three forms

$$g = \begin{pmatrix} \frac{ix_1 + x_2}{z} & \frac{1}{z} \\ -\frac{x_1^2 + x_2^2 + z^2}{z} & \frac{ix_1 - x_2}{z} \end{pmatrix}, \begin{pmatrix} \frac{x_1 + x_2}{z} & \frac{1}{z} \\ -\frac{-x_1^2 + x_2^2 + z^2}{z} & \frac{x_1 - x_2}{z} \end{pmatrix}, \begin{pmatrix} e^{it}\cosh\rho & e^{i\psi}\sinh\rho \\ e^{-i\psi}\sinh\rho & e^{-it}\cosh\rho \end{pmatrix} \tag{3}$$

depending on whether we are considering Euclidean AdS_3, Minkowskian AdS_3 in the Poincaré patch, or global AdS_3 respectively.

Integrability of the AdS_3 σ-model means that there is a family of flat currents parametrized by an arbitrary complex number — the spectral parameter $x \in \mathbb{C}$. Namely defining

$$J = \frac{j}{1 - x} \qquad \bar{J} = \frac{\bar{j}}{1 + x} \tag{4}$$

we find that the equations of motion are equivalent to the flatness condition enforced for arbitrary x:

$$\partial\bar{J} - \bar{\partial}J + [J, \bar{J}] = 0 \tag{5}$$

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[1]We may take these coordinates to be either complex or light-cone depending on the worldsheet signature.

For the following, it will be important to regard the above flatness condition as the compatibility condition for the auxiliary linear problem

$$\partial \Psi + J\Psi = 0$$
$$\bar{\partial} \Psi + \bar{J}\Psi = 0 \tag{6}$$

where $\Psi(w, \bar{w}; x)$ is a 2-component vector. Once one knows two independent solutions of (6), one can put them into two columns of a 2×2 matrix $\hat{\Psi}(w, \bar{w}; x)$ which satisfies the matrix differential equations

$$\partial \hat{\Psi} + J\hat{\Psi} = 0 \qquad \bar{\partial} \hat{\Psi} + \bar{J}\hat{\Psi} = 0 \tag{7}$$

The knowledge of $\hat{\Psi}(w, \bar{w}; x = 0)$ allows us to reconstruct the original string classical solution. Namely, we can at once get the currents from

$$j = -\partial \hat{\Psi} \cdot \hat{\Psi}^{-1}|_{x=0} \tag{8}$$

as well as reconstruct the classical solution by the formula [25]

$$g = \sqrt{\det \hat{\Psi}} \cdot \hat{\Psi}^{-1}|_{x=0} \tag{9}$$

It would seem at first glance that these formulas are not particularly useful, since in order to find $\hat{\Psi}$ directly one would have to solve the system (7) which depends on the knowledge of the classical solution (which is encoded in the currents $J, \bar{J}$). The algebraic curve construction (or 'finite-gap construction') allows, however, to construct $\hat{\Psi}(w, \bar{w}; x)$ directly starting from a given algebraic curve and exploiting general analyticity properties of its dependence on the spectral parameter x. This procedure is described in general in Chapter 5 of [26] and in the context of spinning strings in AdS in [25, 27]. As we will be using it in a quite general form in the present paper, we will review it below.

3 A brief review of the finite gap (algebraic curve) construction

In the context of strings in $AdS_5 \times S^5$, the algebraic curve construction has been adopted exclusively for the case of spinning string solutions. These are classical, closed string solutions of the relevant bosonic σ-model.

The starting point of the construction is the monodromy operator associated to the flat currents defined above.

$$\Omega(w_0, \bar{w}_0; x) = Pe^{\int_C Jdw + \bar{J}d\bar{w}} \tag{10}$$

where $(w_0, \bar{w}_0)$ is some reference point on the worldsheet, and C is a curve going from this point once around the cylinder and going back to $(w_0, \bar{w}_0)$. The flatness of the currents implies that the monodromy does not depend on smooth deformations of C (hence, if C were contractible, the resulting monodromy would be trivial). An easy consequence of this path independence is the behaviour of the monodromy w.r.t. a change of the reference point:

$$\Omega(w_1, \bar{w}_1; x) = U \, \Omega(w_0, \bar{w}_0; x) \, U^{-1} \tag{11}$$

Hence the eigenvalues of the monodromy operator do not depend on the reference point and thus are conserved (they may be computed e.g. using any constant time circle on the worldsheet cylinder). These eigenvalues depend on the spectral parameter and so this construction provides an infinite set of conserved quantities. For the case at hand, the eigenvalues can be written as

$$e^{ip(x)}, \; e^{-ip(x)} \tag{12}$$

where $p(x)$ is the so-called pseudomomentum. The crucial input for the algebraic curve classification of the finite-gap solutions are the analytic properties of $p(x)$ as a function of the spectral parameter.

Let us note at this stage, that it would seem that the whole algebraic curve method would be inapplicable for Wilson loop solutions, for which all loops are contractible and hence the monodromy is trivial – so there is no pseudomomentum to start with. In this paper we will show that in fact one can adopt the algebraic curve classification method to Wilson loops and we will show explicitly how one can associate algebraic curves to certain standard examples and, conversely, how one can explicitly reconstruct these solutions algebraically starting from the given algebraic curves.

An algebraic curve can be constructed out of the monodromy operator by defining

$$L(w, \bar{w}; x) = -i \frac{\partial}{\partial x} \log \Omega(w, \bar{w}; x) \tag{13}$$

which is a 2×2 matrix with rational coefficients and then defining

$$\det(\tilde{y} \cdot 1 - L(w, \bar{w}; x)) = 0 \tag{14}$$

which clearly only depends on $p'(x)$. Redefining $\tilde{y}$ to get rid of double poles at $x = \pm 1$ (see [25] for a discussion) gives the standard genus g algebraic curve Σ of the form

$$y^2 = \prod_{i=1}^{2g+2} (x - a_i) \tag{15}$$

together with a meromorphic differential form dp on Σ, which satisfies a set of conditions (see [7, 8, 9, 10]) allowing e.g. for computing the energy of the classical solution in terms of conserved charges (spins, angular momenta). We are not presenting these expressions here, as we will not use them in the following.

Let us mention a subtlety associated with the algebraic curve of the AdS_3 σ-model. For the commonly studied case of $AdS_3 \times S^1$ [8], the pseudomomentum form dp has double poles at $x = \pm 1$ associated to the J charge of the S^1:

$$p(x) \sim \frac{\pi \frac{J}{\sqrt{\lambda}}}{x \pm 1} + \ldots \tag{16}$$

When $J = 0$ and the solution is completely contained in AdS_3 (i.e. satisfies Virasoro constraints there), $x = \pm 1$ become *branch points* of the algebraic curve Σ and e.g. for the GKP folded string solution at $J = 0$, dp has the form

$$dp = \frac{Ax^2 + B}{(x^2 - 1)\sqrt{(x^2 - 1)(x^2 - a^2)}} dx \tag{17}$$

with an algebraic curve $y^2 = (x^2 - 1)(x^2 - a^2)$. See appendix B.2 for a discussion of the reconstruction of the GKP solution from this algebraic curve.

Reconstruction of the classical solutions from algebraic curves

Let us now briefly sketch how to reconstruct the full classical solution from an algebraic curve for the spinning string introduced above. We will review the reconstruction procedure specializing initially to the spinning string context and indicating, at the end, the passage to the most general case of [26].

The monodromy operator (10) is just the parallel transport *of the solutions of the linear system* (6) around a cycle which goes once around the worldsheet cylinder. So given a fixed point $(w, \bar{w})$ on the worldsheet we will have two distinguished solutions of (6) which will be the eigenvectors of $\Omega(w, \bar{w}; x)$ corresponding to $e^{\pm ip(x)}$. The algebraic curve Σ can be understood as encoding the information how these solutions depend on the spectral parameter x (keeping the reference point $(w, \bar{w})$ fixed). In particular the two branches of the curve above x correspond to these two solutions. More precisely, both solutions, as functions of x, can be described by a *single* vector-valued function on Σ. By abuse of notation, we will write $x \in \Sigma$ when we mean either of the two points in Σ lying above x. These two points will be denoted explicitly by x^+ and x^-.

Reconstruction starts from the realization that Ω can be simultaneously diagonalized with the linear operators $\partial + J$ and $\bar{\partial} + \bar{J}$. Hence a solution of (6) should be proportional to the eigenvector of Ω, the determination of which is a somewhat simpler problem. The eigenvector of Ω can be normalized as

$$\Omega \Psi_n(w, \bar{w}; x) = e^{ip(x)} \Psi_n(w, \bar{w}; x) \quad \text{with} \quad \Psi_n(w, \bar{w}, x) = \begin{pmatrix} 1 \\ \psi_n(w, \bar{w}; x) \end{pmatrix} \tag{18}$$

which defines a single scalar function $\psi_n(w, \bar{w}; x)$. $\psi_n(w, \bar{w}; x)$ is a meromorphic function on Σ and, for a genus g curve Σ, *typically* has $g + 1$ poles.[2] Moreover g of these poles will move on Σ as we change the worldsheet point $(w, \bar{w})$. These are called *dynamical poles*. In order to proceed further, one writes the most general form of $\psi_n(w, \bar{w}; x)$ consistent with these assumptions.

In the second step of the reconstruction procedure, we use the fact that a solution of (6) should be proportional to the above eigenvector:

$$\Psi(w, \bar{w}; x) = f_{BA}(w, \bar{w}; x) \cdot \Psi_n(w, \bar{w}; x) \tag{19}$$

The function $f_{BA}(w, \bar{w}; x)$ is called a Baker-Akhiezer function on Σ (treated as a function of x). It has to satisfy certain analyticity conditions, in particular it should

1. vanish at the dynamical poles

2. have an essential singularity of a prescribed form at the special points $x = \pm 1$ (see [25])

$$f_{BA}(w, \bar{w}; x) \sim e^{const \cdot \frac{w}{x-1}} \qquad f_{BA}(w, \bar{w}; x) \sim e^{const \cdot \frac{\bar{w}}{x+1}} \tag{20}$$

3. as $x \to \infty$, $\Psi(w, \bar{w}; x)$ should become independent of the worldsheet coordinates.

Let us note that it is only in the last two conditions above, that the worldsheet coordinates enter explicitly. Remarkably enough, all these conditions are enough to allow one to reconstruct the full $w, \bar{w}$ and x dependence of the solution of the linear system (6), and thus the original classical string solution through (8)-(9).

Let us note that the reconstruction procedure, as sketched here following [26], does not really depend too much on the fact that we used the

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[2]See Proposition on p. 133 of [26], which rests, however, on some genericity assumptions. We will encounter later an important example where this is violated.

monodromy operator $\Omega(w, \bar{w}; x)$. We could have, and probably should have, used instead its logarithmic derivative $L(w, \bar{w}; x)$. But in fact, for the whole procedure to work, we also do not need to use the specific construction of $L(w, \bar{w}; x)$. What is enough is that it is a Lax operator, i.e. a 2×2 matrix satisfying

$$\partial L + [J, L] = 0$$
$$\bar{\partial} L + [\bar{J}, L] = 0 \tag{21}$$

whose entries are *rational* (or polynomial) functions of x. However, once we make this generalization, we will have to generalize and rederive the condition for the essential singularity (20) of the Baker-Akhiezer function.

4   Questions

After this brief review of the classical algebraic curve approach to classical spinning string solutions we are ready to formulate the key questions which were a motivation for this work.

Question 1. The classical algebraic curve approach in the $AdS_5 \times S^5$ context has been applied for spinning strings, where it started from the notion of the pseudomomentum associated to the monodromy along noncontractible loops. We would like to ask whether one can adapt this framework to describe the classical solutions associated to Wilson loop expectation values. For these solutions there are (typically) no noncontractible loops, so the starting point of the preceding construction does not even exist.

Question 2. Recently, a quite different family of classical string solutions, began to be considered. These are classical solutions corresponding to multi-point correlation functions of $\mathcal{N} = 4$ SYM operators associated with classical spinning strings. These solutions have the topology of a punctured sphere and the external states may be identified with concrete classical spinning strings by requiring that the monodromy around a given puncture coincides *exactly* with the monodromy (pseudomomentum) $p(x)$ of a given classical spinning string solution. This obvious fact is very surprising taking into account the folklore that the space of string solutions with a given pseudomomentum is finite dimensional (e.g. these solutions should be parametrized just by g positions of the dynamical poles plus some finite data etc.). However since the monodromy around each puncture is characterized *exactly* by the pseudomomentum $p_i(x)$ of the corresponding spinning string, the classical solution associated to a correlation function with this given operator will also be described by *the same* algebraic curve as the corresponding finite-gap spinning string solution. Hence the class of solutions with a

given monodromy should be much richer than naively expected. How is this possible? Even more so, there should exist classical string solutions which should be simultaneously associated with three or more distinct algebraic curves! We will not attempt here to address this problem in full generality, but rather study a correlation function of a Wilson loop with a local operator which exhibits similar phenomena (namely identical pseudomomentum with the original spinning string).

The strategy. Recall that the algebraic curve construction for a curve of genus g implies a very particular dependence of the solution of the linear system (6) as a function of the spectral parameter x. Our approach to the above questions is to study, for some specific examples, the associated solutions of (6) and see whether the analytic structure of these explicit solutions implies the existence of a hidden algebraic curve. We will do it for the null cusp and $q\bar{q}$ potential Wilson loop minimal surfaces and for a correlator of a circular Wilson loop with a local operator. We will identify the relevant algebraic curves and show that, based on this information alone, one may reconstruct the original Wilson loop solutions.

5   Examples

In this section we will introduce our basic examples: the light-like cusp Wilson loop solution, the $q\bar{q}$ potential minimal surface and the correlation function of a circular Wilson loop with the $\operatorname{tr} Z^{J}$ operator. In each case we will explicitly write the classical solution, give two independent solutions of the associated linear system (6), and identify an associated algebraic curve through an explicit construction of a *polynomial* Lax matrix. In the following section we will show how starting just with that algebraic curve we may reconstruct the original Wilson loop/correlation function solution.

The null cusp Wilson loop

The null cusp is an Euclidean minimal surface embedded in the Poincaré patch of Minkowskian AdS_3. It is given explicitly as [20, 28]

$$
\begin{aligned}
t &= e^{-\sqrt{2}\sigma} \cosh \sqrt{2}\tau \\
x &= -e^{-\sqrt{2}\sigma} \sinh \sqrt{2}\tau \\
z &= \sqrt{2} e^{-\sqrt{2}\sigma}
\end{aligned}
\tag{22}
$$

with τ, σ coordinates related to the w, $\bar{w}$ ones through $w = \sigma + i\tau$, $\bar{w} = \sigma - i\tau$. The solution is defined on the whole complex plane, so all loops are con-

tractible. The minimal surface approaches the boundary when $\sigma \to +\infty$. Then the two null lines forming the cusp are obtained when one simultaneously takes $\tau \to \pm\infty$.

The two independent solutions of the linear problem (6) take the following explicit form

$$\Psi_1(w,\bar{w};x) = e^{-\frac{1-i}{4}\sqrt{2}\left(iw\sqrt{\frac{1+x}{1-x}}+\bar{w}\sqrt{\frac{1-x}{1+x}}\right)} \begin{pmatrix} e^{\frac{1+i}{4}\sqrt{2}(-iw+\bar{w})} \\ e^{-\frac{1+i}{4}\sqrt{2}(-iw+\bar{w})}\left(-ix+\sqrt{1-x^2}\right) \end{pmatrix}$$
(23)

and

$$\Psi_2(w,\bar{w};x) = e^{\frac{1-i}{4}\sqrt{2}\left(iw\sqrt{\frac{1+x}{1-x}}+\bar{w}\sqrt{\frac{1-x}{1+x}}\right)} \begin{pmatrix} e^{\frac{1+i}{4}\sqrt{2}(-iw+\bar{w})} \\ e^{-\frac{1+i}{4}\sqrt{2}(-iw+\bar{w})}\left(-ix-\sqrt{1-x^2}\right) \end{pmatrix}$$
(24)

These solutions have at least part of the structure which is reminiscent of an underlying algebraic curve. The $\sqrt{1-x^2}$ makes a prominent appearance, the two solutions differ by choosing a different branch of the square root, which, as mentioned in section 3, is characteristic of treating the linear solution as a single function on the two branches of the algebraic curve. Finally the exponential prefactor is suggestive of a Baker-Akhiezer origin, although its singularity does not look at first glance as an isolated essential singularity.

In order to unambiguously associate an algebraic curve with this solution, we will find a *polynomial* Lax matrix $L(w,\bar{w},x)$, i.e. a 2×2 matrix with polynomial dependence on the spectral parameter x, satisfying

$$\partial L + [J, L] = 0 \qquad\qquad \bar{\partial}L + [\bar{J}, L] = 0 \qquad\qquad (25)$$

It is clear that we can solve the above equation by taking *any* expression of the form

$$L(w,\bar{w},x) = \hat{\Psi}(w,\bar{w};x) \cdot A(x) \cdot \hat{\Psi}(w,\bar{w};x)^{-1} \qquad (26)$$

where $\hat{\Psi}$ is a matrix whose columns are any two independent solutions of (6) and $A(x)$ is an arbitrary x-dependent matrix. In general the result will not be a polynomial in x. However, for the case at hand, putting Ψ_1 and Ψ_2 as columns of $\hat{\Psi}$ and taking $A(x)$ to be $A(x) = \sqrt{1-x^2}\,\mathrm{diag}(1,-1)$ gives the following polynomial Lax matrix

$$L(w,\bar{w},x) = \begin{pmatrix} ix & e^{\frac{1+i}{\sqrt{2}}(-iw+\bar{w})} \\ e^{-\frac{1-i}{\sqrt{2}}(w+i\bar{w})} & -ix \end{pmatrix} \qquad (27)$$

Now the algebraic curve is defined by $\det(y - L(w,\bar{w};x)) = 0$, which gives

$$y^2 = 1 - x^2 \qquad\qquad (28)$$

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This is a genus-0 algebraic curve. The reader might be worried that by inserting the $\sqrt{1-x^2}$ factor into $A(x)$ we have put in the answer (28) by hand. This is not so, since the factor $\sqrt{1-x^2}$ was crucial in order to have a *polynomial* Lax matrix.

The elementary solutions (23) and (24) are, by construction, eigenvectors of $L(w, \bar{w}; x)$, i.e.

$$L\,\Psi_1 = \sqrt{1-x^2}\,\Psi_1 \qquad\qquad L\,\Psi_2 = -\sqrt{1-x^2}\,\Psi_2 \qquad (29)$$

It will be important in the following that the Lax matrix is diagonal (with distinct eigenvalues) as $x \to \infty$.

In section 6, we will show that one can explicitly construct $\Psi_{1,2}$, and hence the original classical solution (22) in a completely standard way starting just from the algebraic curve (28).

The $q\bar{q}$ potential Wilson loop

The $q\bar{q}$ Wilson loop minimal surface, introduced in [29, 30], approaches the boundary at two lines at a spacelike separation L. For our purposes, we will need a conformally flat worldsheet parametrization which was first obtained in [31]:

$$z = z_0\,\mathrm{cn}\,\sigma$$
$$x_1 \equiv t = z_0\tau/\sqrt{2}$$
$$x_2 \equiv x = z_0 F(\sigma)/\sqrt{2} \qquad (30)$$

where

$$F(\sigma) = 2E(\mathrm{am}\,\sigma|\tfrac{1}{2}) - \sigma, \qquad (31)$$

E is the incomplete elliptic integral of the second kind,

$$z_0 = \frac{\Gamma(\tfrac{1}{4})^2}{(2\pi)^{\frac{3}{2}}}L \qquad (32)$$

is the maximum bulk extension of the surface (attained at $\sigma = 0$), and the Jacobi amplitude am and Jacobi elliptic functions $\mathrm{cn}, \mathrm{sn}, \mathrm{dn}$ are always taken with a parameter $\tfrac{1}{2}$, i.e. $\mathrm{am}\,\sigma \equiv \mathrm{am}(\sigma|\tfrac{1}{2})$ etc. (for more information on these, see appendix A). $w, \bar{w}$ are defined identically as in the case of the null cusp, but the solution is defined now only on a strip where $\mathrm{cn}\,\sigma \geq 0$, i.e. $|\sigma| \leq K(\tfrac{1}{2})$. At the ends of this interval the worldsheet forms the two parallel Wilson lines on the boundary.

Starting from the Euclidean signature form of g, one proceeds essentially in the same fashion as in the case of cusp, albeit with significant computational complications arising due to the special functions involved. A basis of independent solutions can be taken as

$$\Psi_1 = \frac{\mathcal{E}_+ \sqrt{1 - x \operatorname{cn}^2 \sigma}}{\operatorname{cn} \sigma} \begin{pmatrix} 1 \\ \frac{\sqrt{2} x \operatorname{cn} \sigma \operatorname{sn} \sigma \operatorname{dn} \sigma + i \sqrt{x} \sqrt{1-x^2} \operatorname{cn}^2 \sigma}{x \operatorname{cn}^2 \sigma - 1} - \frac{i\tau + F(\sigma)}{\sqrt{2}} \end{pmatrix} \tag{33}$$

$$\Psi_2 = \frac{i}{\sqrt{x}} \frac{\mathcal{E}_- \sqrt{1 - x \operatorname{cn}^2 \sigma}}{\operatorname{cn} \sigma} \begin{pmatrix} 1 \\ \frac{\sqrt{2} x \operatorname{cn} \sigma \operatorname{sn} \sigma \operatorname{dn} \sigma - i \sqrt{x} \sqrt{1-x^2} \operatorname{cn}^2 \sigma}{x \operatorname{cn}^2 \sigma - 1} - \frac{i\tau + F(\sigma)}{\sqrt{2}} \end{pmatrix} \tag{34}$$

where

$$\mathcal{E}_\pm = \exp \frac{-i\sigma + \tau x + i(1 + x)\Pi(\frac{x}{x-1}; \operatorname{am} \sigma | \frac{1}{2})}{\pm \sqrt{2} \sqrt{x} \sqrt{1 - x^2}} \tag{35}$$

(with Π being the incomplete elliptic integral of the third kind) are the Baker-Akhiezer-like prefactors and we also notice that everything that (essentially) discerns both solutions are different signs of the square root terms. The proportionality constant $i/\sqrt{x}$ is essential to ensure that in the limit $x \to 0$ the matrix $\hat{\Psi}$ will be invertible and its determinant positive.

Let us note that the above solution has one feature which naively excludes the possibility of an underlying algebraic curve – the factor

$$\sqrt{1 - x \operatorname{cn}^2 \sigma} \tag{36}$$

This would indicate the existence of a branch cut whose position is dependent on the worldsheet coordinate, which is at odds with any kind of algebraic curve description. One finds, however, that this branch cut is cancelled by a corresponding cut in $\mathcal{E}_\pm$.

We then construct a polynomial Lax matrix choosing $\hat{\Psi} = (\Psi_1 \Psi_2)$ and $A(x) = \sqrt{x}\sqrt{1 - x^2} \cdot \operatorname{diag}(1, -1)$. The result is definitely more complicated than in the previous case, but the characteristic polynomial is nonetheless simple and the algebraic curve is in this case defined by

$$y^2 = x(1 - x^2). \tag{37}$$

This is an elliptic (genus-1) curve. The eigenvalues associated to $\Psi_{1,2}$ as eigenvectors of L are

$$L \Psi_1 = \sqrt{x}\sqrt{1 - x^2}\, \Psi_1 \qquad L \Psi_2 = -\sqrt{x}\sqrt{1 - x^2}\, \Psi_2. \tag{38}$$

A significant complication in the present case is the fact that $x = \infty$ is a branch point of the algebraic curve (37). Consequently, the asymptotics of $L(w, \bar{w}; x)$ as $x \to \infty$ are more subtle:

$$L(w, \bar{w}; x) \propto x^2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \dots \tag{39}$$

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The $\langle W(C)\mathrm{tr}\, Z^J \rangle$ correlation function

Let us now consider a classical solution which corresponds to a correlation function of a circular Wilson loop with the local operator $\mathrm{tr}\, Z^J$ (the BMN vacuum). This example is interesting for a different reason than the previous two. Now we have a noncontractible loop and nontrivial monodromy, but that monodromy is completely determined by the pseudomomentum of the local operator – here that of the BMN vacuum namely:

$$p(x) = \frac{2\pi j x}{x^2 - 1} \tag{40}$$

where $j = J/\sqrt{\lambda}$. This pseudomomentum is identical to the one in a classical configuration corresponding to a correlation function of two local operators

$$\left\langle \mathrm{tr}\, \bar{Z}^J \mathrm{tr}\, Z^J \right\rangle \tag{41}$$

(or its global AdS counterpart – the standard BMN pointlike string). What distinguishes these two configurations? Clearly the operator $\mathrm{tr}\, Z^J$ may appear in arbitrarily complicated correlation functions, yet all of them will have exactly the same monodromy (40).

We will contrast here the two cases: $\left\langle \mathrm{tr}\, \bar{Z}^J \mathrm{tr}\, Z^J \right\rangle$ and $\left\langle W(C)\mathrm{tr}\, Z^J \right\rangle$. By a special conformal transformation, one can always put the coordinate of the insertion point of the local operator $\mathrm{tr}\, Z^J$ to infinity (equivalently the string goes to the center of AdS). The second operator will be at the origin, or in the case of the Wilson loop, the loop will be a unit circle around the origin.

In the first case (two local operators) the transformed classical solution is just

$$z = e^{j\tau} \qquad \phi = ij\tau \tag{42}$$

In the second case (a local operator and the circular Wilson loop) the relevant solution has been found by Zarembo in [18]:

$$\begin{aligned}
x_1 &= \frac{\sqrt{1+j^2}\, e^{j\tau}}{\cosh(\sqrt{1+j^2}\, \tau + \xi)} \cos\sigma \\
x_2 &= \frac{\sqrt{1+j^2}\, e^{j\tau}}{\cosh(\sqrt{1+j^2}\, \tau + \xi)} \sin\sigma \\
z &= \left(\sqrt{1+j^2}\, \tanh\left(\sqrt{1+j^2}\, \tau + \xi \right) - j \right) e^{j\tau} \\
\phi &= ij\tau
\end{aligned} \tag{43}$$

where

$$\xi = \log\left(j + \sqrt{1+j^2} \right) \tag{44}$$

The solution of the linear system is very simple for the first case. It is given by

$$\Psi_1(w,\bar{w};x) = e^{-\frac{j}{2(1-x)}w - \frac{j}{2(1+x)}\bar{w}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \Psi_2(w,\bar{w};x) = e^{\frac{j}{2(1-x)}w + \frac{j}{2(1+x)}\bar{w}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
(45)

where $w = \tau + i\sigma$ and $\bar{w} = \tau - i\sigma$. A Lax matrix can be constructed immediately with $A(x) = \mathrm{diag}(1,-1)$. The resulting algebraic curve is just

$$y^2 = 1 \tag{46}$$

This means that we have just two disconnected copies of the complex plane (or the sphere).

The case of the Wilson loop correlation function is by contrast much more complicated. The solutions of the linear system are

$$\Psi_1 = \frac{e^{-\frac{j}{2(1-x)}w - \frac{j}{2(1+x)}\bar{w}}}{e^{\sqrt{1+j^2}\,(w+\bar{w})} - 1} \begin{pmatrix} \frac{-i}{2\sqrt{1+j^2}}(e^{\sqrt{1+j^2}\,(w+\bar{w})}-1)\left(x-j-\sqrt{1+j^2}\left(1+\frac{2}{e^{\sqrt{1+j^2}\,(w+\bar{w})}-1}\right)\right) \\ e^{\frac{1}{2}(-1+j+\sqrt{1+j^2})w}e^{\frac{1}{2}(1+j+\sqrt{1+j^2})\bar{w}} \end{pmatrix}$$

$$\Psi_2 = \frac{e^{\frac{j}{2(1-x)}w + \frac{j}{2(1+x)}\bar{w}}}{e^{\sqrt{1+j^2}\,(w+\bar{w})} - 1} \begin{pmatrix} e^{\frac{1}{2}(1-j+\sqrt{1+j^2})w}e^{\frac{1}{2}(-1-j+\sqrt{1+j^2})\bar{w}} \\ \frac{-i}{2\sqrt{1+j^2}}(e^{\sqrt{1+j^2}\,(w+\bar{w})}-1)\left(x-j+\sqrt{1+j^2}\left(1+\frac{2}{e^{\sqrt{1+j^2}\,(w+\bar{w})}-1}\right)\right) \end{pmatrix}$$
(47)

Apart from being much more complicated, the above expressions are quite surprising. Firstly, we see that once we would normalize the vector by keeping the upper component equal to 1, the position of the pole would move depending on the point of the worldsheet – it would be a dynamical pole. Yet, the pseudomomentum is trivial and characteristic of a simple point-like string associated to a genus-0 curve and thus with no dynamical poles.

Let us now construct the Lax matrix and identify the corresponding algebraic curve. It turns out that a *polynomial* Lax matrix can be constructed by taking $A(x) = (1+2jx-x^2)\cdot\mathrm{diag}(1,-1)$. The expression for the resulting Lax matrix is quite involved, but yields the relatively simple algebraic curve

$$y^2 = (1 + 2jx - x^2)^2 \tag{48}$$

We see a new feature appearing – double zeroes on the r.h.s. So there are no true branch cuts but rather degeneracies of the curve. We will show in the following section that these degeneracies play a crucial role in reconstructing the solutions (47) and hence also (43). Indeed it is worth pointing out that the pseudomomentum $p(x)$ is not necessarily enough to completely characterize an algebraic curve.

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6 Reconstructing the solutions from algebraic curves

In this section we will show how to reconstruct the solutions of the linear system (6) from the algebraic curves identified in the previous section.

Since we will assume that the algebraic curves came from quite generic polynomial Lax matrices which were not associated to any kind of monodromy, we have to rederive, following [26], the conditions for the essential singularity of the Baker-Akhiezer function.

The starting point is the very general fact ([26, eq. (3.15)]), that the flat currents J and $\bar{J}$ can be extracted by taking the *singular* terms in the Laurent expansion of some polynomial in L with coefficients being rational functions of x, i.e.

$$[P(L(w, \bar{w}; x), x)]_{x=1}^- = J(w, \bar{w}; x) \tag{49}$$

and similarly at $x = -1$ for $\bar{J}$ (taking possibly a different polynomial).

For the case of the AdS_3 σ-model and the studied solutions, this general rule simplifies dramatically and we always have

$$\left[\frac{c_1}{1-x} L(w, \bar{w}; x)\right]_{x=1}^- = J(w, \bar{w}; x) \qquad \left[\frac{c_{-1}}{1+x} L(w, \bar{w}; x)\right]_{x=-1}^- = \bar{J}(w, \bar{w}; x) \tag{50}$$

The constants in the above formula are arbitrary and can be changed by a linear redefinition[3] of the worldsheet coordinates w and $\bar{w}$. They may indeed be complex (and not neccessarily complex conjugate to each other) which then serves to pick the wanted signature of the worldsheet, i.e. to use light-cone or holomorphic coordinates.

Once we have (50), the conditions for the essential singularity around $x = \pm 1$ of the solutions of the linear system (6) directly follow [26]. Indeed we can rewrite $\partial \Psi + J \Psi = 0$ as

$$\partial \Psi + \frac{c_1}{1-x} L \cdot \Psi + \text{regular} \cdot \Psi = 0 \tag{51}$$

The second term is very simple, since we are interested in solutions which are eigenvectors of L. So we may substitute $L \cdot \Psi$ by $y(x)\Psi$ where $y(x)$ follows from the algebraic curve associated to L. Now around $x = 1$, we can drop the last term and obtain the behaviour

$$\Psi \sim e^{-\frac{c_1 y(x)}{1-x} w - \frac{c_{-1} y(x)}{1+x} \bar{w}} \cdot \text{regular} \tag{52}$$

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[3]In fact the constants could be also generalized to arbitrary holomorphic and anti-holomorphic functions $c_1(w)$, $c_{-1}(\bar{w})$ through a conformal redefinition of the worldsheet coordinate.

It is important to emphasize that the above asymptotics is a priori valid *only* in the neighborhood of $x = \pm 1$. We will see specific examples below.

The null cusp Wilson loop — reconstruction

For the null cusp we start from the algebraic curve

$$y^2 = 1 - x^2 \tag{53}$$

First let us fix the constants in (50). We could have just as well left these constants arbitrary and redefined the worldsheet coordinates at the end of the calculation. Here, for simplicity, we will just substitute the values corresponding to (23)-(24) from the start. We find that

$$c_1 = \frac{1+i}{2\sqrt{2}} \qquad c_{-1} = \frac{1-i}{2\sqrt{2}} \tag{54}$$

Now the asymptotics (52) yields

$$e^{-\left[\frac{1+i}{2\sqrt{2}}\sqrt{\frac{1+x}{1-x}}w + \frac{1-i}{2\sqrt{2}}\sqrt{\frac{1-x}{1+x}}\bar{w}\right]} \tag{55}$$

which exactly coincides with the overall exponential factor in (23) and (24). We will justify the form of this expression away from $x = \pm 1$ more rigorously below.

Let us now perform the reconstruction according to the procedure of section 3. It is first convenient to uniformize the algebraic curve $y^2 = 1 - x^2$ by the parametrization

$$y = \frac{2t}{1+t^2} \qquad x = \frac{1-t^2}{1+t^2} \tag{56}$$

In this way we get rid of all ambiguous cuts in our expressions. Passing to the other sheet corresponds to the transformation $t \to -t$. The points above $x = \infty$, namely $x = \infty^+$ and $x = \infty^-$ correspond to the points $t = \pm i$. The point $x = 1$ corresponds to $t = 0$, while $x = -1$ corresponds to $t = \infty$.

We will first determine the normalized eigenvector of L (without using, of course, the specific form of $L(w, \bar{w}; x)$ but only very general properties like the diagonalizability at $x \to \infty$)

$$\Psi_n(w, \bar{w}; x) = \begin{pmatrix} 1 \\ \psi_n(w, \bar{w}; x) \end{pmatrix} \tag{57}$$

Since the genus of the algebraic curve is zero, we expect the function $\psi_n(w, \bar{w}; x)$ to have just a single pole. At $x = \infty$ the Lax matrix is diagonal (can be diagonalized), so we should have

$$\Psi_n(w, \bar{w}; x = \infty^+) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \Psi_n(w, \bar{w}; x = \infty^-) = \begin{pmatrix} 1 \\ \infty \end{pmatrix} \qquad (58)$$

This is enough to fix completely the spectral parameter dependence of $\Psi_n(w, \bar{w}, x)$:

$$\Psi_n(w, \bar{w}; t) = \begin{pmatrix} 1 \\ a(w, \bar{w}) \frac{t-i}{t+i} \end{pmatrix} \equiv \begin{pmatrix} 1 \\ -a(w, \bar{w}) \left(x + i\sqrt{1 - x^2} \right) \end{pmatrix} \qquad (59)$$

We see, that we have recovered the vector structure of (23)-(24).

Now it remains to determine the Baker-Akhiezer function. We have already fixed the essential singularities. Now we can justify why the expressions in the exponent of (55) are correct not only in the neighborhood of $x = \pm 1$ but in fact for all x. Indeed

$$\sqrt{\frac{1+x}{1-x}} = \frac{1}{t} \qquad \sqrt{\frac{1-x}{1+x}} = t \qquad (60)$$

so these are the unique functions on the algebraic curve which have only a single pole at $t = 0$ ($x = 1$) and at $t = \infty$ ($x = -1$). Since there are no dynamical poles[4] in $\Psi_n(w, \bar{w}; x)$, the whole x dependence of the Baker-Akhiezer function is now fixed. So currently we have

$$\Psi(w, \bar{w}; t) = f_{BA}(w, \bar{w}; t) \cdot \Psi_n(w, \bar{w}; t)$$
$$= e^{-\left[\frac{1+i}{2\sqrt{2}} \frac{1}{t} w + \frac{1-i}{2\sqrt{2}} t \bar{w} \right]} b(w, \bar{w}) \cdot \begin{pmatrix} 1 \\ a(w, \bar{w}) \frac{t-i}{t+i} \end{pmatrix} \qquad (61)$$

It remains to fix the functions $a(w, \bar{w})$ and $b(w, \bar{w})$. Remarkably enough this can be done using the obvious property that $\Psi(w, \bar{w}, x)$ becomes w and $\bar{w}$ independent when $x \to \infty$. This follows from the fact that then the flat currents vanish.

In our case we have to enforce this condition both at $x = \infty^+$ and at $x = \infty^-$.

At $x = \infty^+$ ($t = i$) we should impose this condition on the top component of $\Psi(w, \bar{w}; t)$ and find $b(w, \bar{w})$:

$$b(w, \bar{w}) = \frac{1}{f_{BA}(w, \bar{w}; t = i)} = e^{\frac{1+i}{2\sqrt{2}}(-iw + \bar{w})} \qquad (62)$$

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[4]I.e. poles whose position depends on the worldsheet coordinates w and $\bar{w}$.

At $x = \infty^-$ $(t = -i)$ we should concentrate on the lower component to find $a(w, \bar{w}) = 1/b^2(w, \bar{w})$ which gives for the relevant product

$$a(w, \bar{w})b(w, \bar{w}) = e^{-\frac{1+i}{2\sqrt{2}}(-iw+\bar{w})} \qquad (63)$$

We see at this stage that we have completely recovered the solutions of the linear system (23)-(24), purely from the algebraic curve $y^2 = 1 - x^2$ and some minimal assumptions on the form of L (diagonalizability at $x = \infty$ and the form (50)).

The $q\bar{q}$ potential Wilson loop — reconstruction

In this section we will use elliptic theta functions with a square period lattice with quasiperiods $2K = 2K(\frac{1}{2}), 2iK' = 2iK$ (for the details on notation and properties of the doubly periodic functions, see appendix A). We define the functions $x(z), y(z)$ so that x has a double pole at iK and a double zero over K, while y has zeroes over $0, K, K + iK$ and a triple pole over iK. Thus, up to a multiplicative constant,

$$x(z) \propto \frac{\theta(z - K)\theta(z + K)}{\theta(z - iK)\theta(z + iK)} \qquad (64)$$

$$y(z) \propto \frac{\theta(z)\theta(z - K)\theta(z + K + iK)}{\theta(z - iK)\theta(z + iK)^2}. \qquad (65)$$

They are periodic in both directions. We choose the proportionality constant for x so that $x(0) = 1, x(K + iK) = -1$. Then examining poles and zeroes on both sides of the algebraic curve equation (37)

$$y^2 = x(1 - x^2) \qquad (66)$$

we see that they coincide, so by choosing a proportionality constant for y the above equation can be exactly satisfied. This is thus a parameterization of this algebraic curve, with a property that flipping the sign of z corresponds to passing from any given point to its counterpart on the other sheet of the curve (due to the fact that x, y are even and odd, respectively).

The Baker-Akhiezer prefactor has the following asymptotic structure:

$$f_{BA}(w, \bar{w}; z) = \exp\left\{ \frac{-i}{2\sqrt{2}} \left(\frac{y}{1 - x}w + \frac{-y}{1 + x}\bar{w} \right) \right\} \times \text{regular} \qquad (67)$$

where again the constants were specifically chosen but in principle could have been redefined at the very end. However, we will choose to work with σ, τ

instead of $w = \sigma + i\tau, \bar{w} = \sigma - i\tau$, motivated by the fact that the original solution for this case (33)-(34) was more conveniently expressed in terms of these. We have to ensure that the Baker-Akhiezer prefactor does not have any essential singularities other than $x = \pm 1$ (i.e. $z = 0$ or $z = K + iK$). We see, however, that the exponent in (67) has the following poles:

$$\frac{y}{1-x} = i\frac{\sqrt{2}}{z} - i\frac{\sqrt{2}}{z - iK} + \text{regular}, \tag{68}$$

$$\frac{-y}{1+x} = i\frac{\sqrt{2}}{z - K - iK} - i\frac{\sqrt{2}}{z - iK} + \text{regular}, \tag{69}$$

so in their difference (multiplied by τ in the exponent) the second terms will cancel out. Hence the τ-dependent part becomes

$$\exp\left\{\frac{-i}{2\sqrt{2}}\left(\frac{y}{1-x} + \frac{y}{1+x}\right)i\tau\right\} = \exp\left\{\frac{x\tau}{\sqrt{2}y}\right\} \tag{70}$$

However in the coefficient of σ, the pole at $z = iK$ corresponds to $x = \infty$ and as such is forbidden in the Baker-Akhiezer function properties outlined in section 3. The function multiplying σ in the exponent will have to have a pole at $z = 0$ and $z = K + iK$ with the prescribed residues. Such a function can be explicitly constructed as

$$G(z) = -\tfrac{1}{2}(\phi(z) + \phi(z - K - iK)). \tag{71}$$

where $\phi(z)$ is the logarithmic derivative of $\theta(z)$ (see appendix A).

Now f_{BA} is no longer periodic in the imaginary direction as the residues do not sum up to zero, and to remedy this we supply it with another factor

$$\frac{\theta(z - \gamma(\sigma, \tau))}{\theta(z - \gamma(0, 0))}. \tag{72}$$

$\gamma(\sigma, \tau)$ denotes the position of the dynamical pole (as f_{BA} has to vanish there). Demanding the double periodicity of f_{BA} fixes the position of the dynamical pole to $\gamma = -i\sigma$ (with $\gamma(0, 0) = 0$).

The ansatz for the solution of (6) becomes

$$\Psi = A(\sigma, \tau) \cdot \exp\left(\frac{x\tau}{\sqrt{2}y} + iG(z)\sigma\right) \frac{\theta(z + i\sigma)}{\theta(z)} \begin{pmatrix} 1 \\ \psi(\sigma, \tau; z) \end{pmatrix} \tag{73}$$

where $A(\sigma, \tau)$ is the σ, τ-dependent regular part of f_{BA}. The function ψ should have its poles at $x = \infty$ and at $z = \gamma$ and should be a well defined

function on the elliptic curve (66). It can be constructed in the following form

$$\psi(\sigma, \tau; z) = r_0(\sigma, \tau) + r_1(\sigma, \tau)(\phi(z - iK) - \phi(z + i\sigma)) \qquad (74)$$

where we have chosen the residues at both poles to cancel to ensure periodicity.

We now determine the unknown functions according to the requirement that Ψ be constant at $x = \infty$ and, due to the fact that $x(z)$ has a double pole at $z = iK$, we have to require that Ψ is constant at the two leading orders in the expansion around $z = iK$. Using the expansions

$$f_{BA} = A(\sigma, \tau)(f_0(\sigma, \tau) + (z - iK)f_1(\sigma, \tau) + \ldots), \qquad (75)$$

$$\psi = \frac{\psi_{-1}(\sigma, \tau)}{z - iK} + \psi_0(\sigma, \tau) + \ldots, \qquad (76)$$

we can write

$$\Psi = A(\sigma, \tau) \left(\begin{array}{ccc} 0 & + & f_0(\sigma, \tau) & + \ldots \\ \frac{f_0(\sigma,\tau)\psi_{-1}(\sigma,\tau)}{z-iK} & + & \psi_{-1}(\sigma, \tau)f_1(\sigma, \tau) + f_0(\sigma, \tau)\psi_0(\sigma, \tau) & + \ldots \end{array} \right) \qquad (77)$$

and demand that all the above coefficients be constant at $z = iK$. We obtain the following solution:

$$A(\sigma, \tau) = \frac{C_1}{f_0(\sigma, \tau)} \qquad\qquad r_1(\sigma, \tau) = \psi_{-1}(\sigma, \tau) = \frac{C_2}{C_1} \qquad (78)$$

$$\psi_0(\sigma, \tau) = \frac{C_3 - A(\sigma, \tau)\psi_{-1}(\sigma, \tau)f_1(\sigma, \tau)}{A(\sigma, \tau)f_0(\sigma, \tau)} = \frac{C_3}{C_1} - \frac{f_1(\sigma, \tau)}{f_0(\sigma, \tau)} \cdot \frac{C_2}{C_1} \qquad (79)$$

and r_0 (contained in ψ_0) is then

$$r_0(\sigma, \tau) = \frac{C_3}{C_1} - \frac{f_1(\sigma, \tau)}{f_0(\sigma, \tau)} \cdot \frac{C_2}{C_1} - \frac{C_2}{C_1} \left(\frac{\theta''(0)}{2\theta'(0)} - \phi(iK + i\sigma) \right). \qquad (80)$$

This is of course at first sight very different from (33)-(34), but some agreement is to be expected, firstly due to the fact that the Jacobi and theta functions are related (albeit very intricately). Secondly, we might notice that regardless of the value of z, Ψ ceases to be well defined at $\sigma = \pm K$, due to the factor $\theta(iK + i\sigma)$ that is present in f_0, a denominator of A. This means that the domain of this solution is $\sigma \in (-K, K)$, precisely the same as for the original solution. Note that this actually follows from a specific choice of $\gamma(0, 0)$, as alternative values would shift the domain or lead generically to complex solutions.

Finally, for a specific choice of the constants $C_{1,2,3}$ we get exact agreement. If we choose $C_1 = \sqrt{1-x}, C_2/C_1 = i\sqrt{2}, C_3 = 0$, then the result is numerically equal to (33) (with replacements $x \to x(z), \sqrt{x}\sqrt{1-x^2} \to y(z)$) up to hundreds of decimal digits for all σ, τ, z.

The $\langle W(C)\operatorname{tr} Z^J \rangle$ correlation function — reconstruction

Here we start from the degenerate curve

$$y^2 = (1 + 2jx - x^2)^2 \tag{81}$$

In this case it is not completely obvious what conditions to impose on the analytic structure of the solutions of the linear system. A point of view which we will adopt here will be to consider the curve (81) as a degenerate limit of a curve with two very small cuts. Thus we may treat it as a degenerate limit of an elliptic curve. Fortunately, we do not need to perform the elliptic construction first and only at the end take the limit — we may directly work with the degenerate curve, drawing from the genus-1 case only some very general analyticity properties. However, for this degenerate curve, we cannot rule out the existence of some other different constructions.

Firstly, the two sheets of (81) are completely distinct and there is no analytical continuation between them. Hence we may, and should, consider two separate vector functions for the two independent linear solutions of (6). Secondly, as we may expect the curve to come as a degeneration of an elliptic curve, we expect to have one kinematical pole at $x = \infty$, and one dynamical pole (depending on w and $\bar{w}$). We have to distribute those two poles between the two branches. Thirdly, at the points of degeneration $1 + 2jx - x^2 = 0$, we will require the two solutions to coincide.

Let us start from the essential singularities at $x = \pm 1$. In this case we find the constants to be

$$c_1 = -\frac{1}{4} \qquad\qquad c_{-1} = \frac{1}{4} \tag{82}$$

which gives the behaviour

$$\Psi \sim e^{\frac{1}{4}\frac{y(x)}{1-x}w - \frac{1}{4}\frac{y(x)}{1+x}\bar{w}} \cdot \text{regular} \tag{83}$$

However care must be taken here, since $y(x) = \pm(1 + 2jx - x^2)$. We cannot substitute this full expression into the exponent since this would generate an unwanted essential singularity at $x = \infty$. Hence it is simplest to just

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substitute $y(1)$ in the first term and $y(-1)$ in the second term.[5] We get therefore

$$\Psi \sim e^{\pm\left(\frac{1}{2}\frac{j}{1-x}w+\frac{1}{2}\frac{j}{1+x}\bar{w}\right)} \tag{84}$$

Now we have to distribute the poles among the two solutions. We will put the pole at $x = \infty$ in the first solution and the dynamical pole in the second. This choice leads to the following ansatz:

$$\Psi_1(w,\bar{w};x) = e^{\frac{1}{2}\frac{j}{1-x}w+\frac{1}{2}\frac{j}{1+x}\bar{w}}c_1(w,\bar{w})\left(\begin{array}{c}1\\b_1(w,\bar{w})(x-a_1(w,\bar{w}))\end{array}\right) \tag{85}$$

$$\Psi_2(w,\bar{w};x) = e^{-\frac{1}{2}\frac{j}{1-x}w-\frac{1}{2}\frac{j}{1+x}\bar{w}}c_2(w,\bar{w})(x-a_2(w,\bar{w}))\left(\begin{array}{c}1\\\frac{b_2(w,\bar{w})}{x-a_2(w,\bar{w})}\end{array}\right) \tag{86}$$

Now we impose the condition that at $x \to \infty$, the solution becomes w, $\bar{w}$ independent. This gives the relations $b_1(w,\bar{w}) = 1/c_1(w,\bar{w})$ and $c_2(w,\bar{w}) = 1$. So at this stage our ansatz takes the form

$$\Psi_1(w,\bar{w};x) = e^{\frac{1}{2}\frac{j}{1-x}w+\frac{1}{2}\frac{j}{1+x}\bar{w}}\left(\begin{array}{c}c_1(w,\bar{w})\\x-a_1(w,\bar{w})\end{array}\right) \tag{87}$$

$$\Psi_2(w,\bar{w};x) = e^{-\frac{1}{2}\frac{j}{1-x}w-\frac{1}{2}\frac{j}{1+x}\bar{w}}\left(\begin{array}{c}x-a_2(w,\bar{w})\\b_2(w,\bar{w})\end{array}\right) \tag{88}$$

Finally, since we expect that the two different functions should come from the same function on the (almost degenerate) elliptic curve, we require that at the two points of degeneration

$$x = j \pm \sqrt{1+j^2} \tag{89}$$

we have

$$\Psi_1\left(w,\bar{w};j+\sqrt{1+j^2}\right) = \Psi_2\left(w,\bar{w};j+\sqrt{1+j^2}\right) \tag{90}$$

$$\Psi_1\left(w,\bar{w};j-\sqrt{1+j^2}\right) = \Psi_2\left(w,\bar{w};j-\sqrt{1+j^2}\right) \tag{91}$$

This gives a set of four linear equations for the four unknown functions

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[5]A possible piece proportional to $(x-1)$ or $(x+1)$ would be automatically cancelled later in the calculation.

$a_1(w, \bar{w})$, $a_2(w, \bar{w})$, $b_2(w, \bar{w})$ and $c_1(w, \bar{w})$. The solution is

$$a_1(w, \bar{w}) = j - \sqrt{1 + j^2} \left(1 - \frac{2}{1 - e^{(w + \bar{w})\sqrt{1 + j^2}}} \right) \tag{92}$$

$$a_2(w, \bar{w}) = j + \sqrt{1 + j^2} \left(1 - \frac{2}{1 - e^{(w + \bar{w})\sqrt{1 + j^2}}} \right) \tag{93}$$

$$b_2(w, \bar{w}) = -\frac{2\sqrt{1 + j^2} e^{-\frac{w}{2}(1 - j - \sqrt{1 + j^2}) + \frac{\bar{w}}{2}(1 + j + \sqrt{1 + j^2})}}{1 - e^{(w + \bar{w})\sqrt{1 + j^2}}} \tag{94}$$

$$c_1(w, \bar{w}) = \frac{2\sqrt{1 + j^2} e^{\frac{w}{2}(1 - j + \sqrt{1 + j^2}) - \frac{\bar{w}}{2}(1 + j - \sqrt{1 + j^2})}}{1 - e^{(w + \bar{w})\sqrt{1 + j^2}}} \tag{95}$$

and coincides with the quite intricate expressions (47) for the solution of the linear system.

7 Applications and conclusions

The aim of this paper was to show that the classical algebraic curve (finite-gap) classification of spinning string solutions in $AdS_5 \times S^5$ can be significantly expanded to encompass other more general classes of solutions, namely Wilson loops and, possibly, correlation functions.

The first case is perhaps not surprising from the point of view of the classical literature on minimal surfaces and integrable models [32], although the focus there has been always rather different and the kind of minimal surfaces relevant for computing Wilson loop expectation values within the AdS/CFT correspondence did not appear. However, it definitely points at a new direction in the context of the spinning string classification, as all these Wilson loops have no noncontractible loops, hence no monodromy and no pseudomomentum $p(x)$, whose analytic properties were the starting point for the spinning string classification [7, 8, 9, 10].

In this paper we showed that for certain classical Wilson loop minimal surfaces in AdS_3, namely the one associated with a null cusp and the infinite rectangular Wilson loop responsible for the $q\bar{q}$ potential, there exists an underlying algebraic curve description. We can associate a definite algebraic curve with each of these solutions and conversely, starting just from that algebraic curve, we can reconstruct the explicit target-space form of the classical solution.

These results have, on the one hand, a purely practical application of suggesting new methods of constructing minimal surfaces in an Anti-de-Sitter

spacetime starting from some higher genus algebraic curves. In this respect, it would be very interesting to understand the precise relation (or even perhaps equivalence) with the very interesting constructions of [33]. On the other hand, the main motivation for us was more theoretical, as the finite-gap constructions of spinning strings had a lot in common with Bethe equations and the comparison with analogous constructions for weak coupling spin chains played a very important role in the development of integrability.

It would be very interesting to understand if there is a similar underlying Bethe ansatz interpretation of the Wilson loops with an algebraic curve description, and in particular understand the relation with the very recent works [22, 23].

On a less speculative level, from the perspective of algebraic curves we may understand quite easily the possible limit-like relations between various string solutions. Of particular interest is the very close relation of the null cusp solution with the large spin limit of the GKP folded string [20]. This relation is especially important, as the GKP string is a closed string solution which is describable *at all couplings* by the all-loop Bethe ansatz[6] [34, 35]. Let us see how this relation arises from the point of view of the identified algebraic curves.

The null cusp is described by the curve $y^2 = (x^2 - 1)$. One can make a deformation of the above curve by adding two additional branch points and taking them to infinity. This suggests to consider the curve

$$y^2 = (x^2 - 1)(x^2 - a^2) \tag{96}$$

in the $a \to \infty$ limit. As we show in appendix B.2, this curve is indeed the algebraic curve underlying the GKP folded string.

A natural very interesting question is whether a similar relation exists for the Wilson loop describing the $q\bar{q}$ potential, i.e. whether there exists a (closed string) solution which would approximate in some form the $q\bar{q}$ minimal surface. To this end we should deform the algebraic curve $y^2 = x(x^2 - 1)$. A natural choice would be to use the curve

$$y^2 = (x^2 - 1)(x - a)(x + 1/a) \tag{97}$$

and take the limit $a \to \infty$. In appendix B.3, we identify the corresponding string solution. Unfortunately it turns out to be also a Wilson loop minimal surface – namely the generalized Wilson loop of two parallel lines on the boundary of global AdS_3 with an angular separation. This configuration

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[6]The Bethe ansatz description is valid in the large spin limit. For generic spins, the description would be in terms of TBA/NLIE equations.

indeed has been proposed in [21] as a generalization of the ordinary $q\bar{q}$ potential and used very recently in [22, 23]. Unfortunately we do not find a closed string counterpart. However, we cannot rule out that some complexified version with fine-tuned parameters (or some genus-2 degeneration) could exist.

The second line of generalization of the classical finite-gap constructions is the case of correlation functions with a local operator. For these classical solutions, the monodromy around the puncture where the local operator would be inserted should be, by definition, *identical* to the monodromy of the corresponding spinning string. Hence the ordinary classical algebraic curve which is constructed out of the pseudomomentum would be identical to the one for the spinning string.

Yet clearly, there is a multitude of nonvanishing correlation functions in which even the simplest operator like the BMN vacuum $\text{tr}\, Z^J$ could participate. This suggests that the space of solutions with given pseudomomentum around a puncture should be extremely vast. This is in a naive contradiction with the folklore that the space of classical solutions of a genus-g algebraic curve is finite dimensional.

We address this problem by examining a simple example of a correlation function of the circular Wilson loop with the BMN operator $\text{tr}\, Z^J$. We find that even though the pseudomomentum is the same, the algebraic curve constructed from a polynomial Lax matrix is singular and can be treated as a degeneration of an elliptic curve. The singularities play a key role in the reconstruction of the Wilson loop correlator from the algebraic curve. Intuitively, the solution may be understood as a soliton (degenerate cuts) on top of a finite-gap spinning string. It would be interesting to explore these types of constructions for other local operators/spinning string solutions.

Clearly, in the case of correlation functions this result is just scratching the surface. For 3-point correlation functions, we expect the classical string solutions to be *simultaneously* describable by three distinct algebraic curves, even of different genera. Currently, we do not possess even a single example (even in some simplified integrable model) with such characteristics. It would be very interesting to understand the structure of such solutions from the algebraic curve perspective.

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A   Useful elliptic functions

In this appendix we will review certain basic properties of both Jacobi elliptic functions and theta functions. We will largely limit the scope to the properties essential to our calculations; for a more comprehensive discussion, including different notations encountered in the literature, refer eg. to the relevant chapters of [36] or [37].

The **Jacobi elliptic functions** are defined (in one of many equivalent ways) as follows: if

$$u = F(\varphi|m) = \int_0^\varphi \frac{\mathrm{d}\theta}{\sqrt{1 - m\sin^2\theta}} \tag{98}$$

where $F(\varphi|m)$ is the incomplete elliptic integral of the first kind, then

$$\mathrm{am}\, u = \varphi \qquad\qquad \mathrm{sn}\, u = \sin\varphi \tag{99}$$

$$\mathrm{cn}\, u = \cos\varphi \qquad\qquad \mathrm{dn}\, u = \sqrt{1 - m\sin^2\varphi} \tag{100}$$

where the first function is called the (Jacobi) amplitude. The number m is a second, usually suppressed, argument to all of the functions, called parameter (as opposed to an alternative notation which uses its square root, called modulus, instead).

Among the elementary properties of the Jacobi elliptic functions are the relations between their square roots (directly following from the above definitions and the trigonometric unity):

$$\mathrm{sn}^2 u + \mathrm{cn}^2 u = m\,\mathrm{sn}^2 u + \mathrm{dn}^2 u = 1. \tag{101}$$

Essential to our derivations are their derivatives (with respect to the non-suppressed argument) as well:

$$\mathrm{sn}' u = \mathrm{cn}\, u \,\mathrm{dn}\, u \qquad \mathrm{cn}' u = -\,\mathrm{sn}\, u \,\mathrm{dn}\, u \qquad \mathrm{dn}' u = -m\,\mathrm{sn}\, u \,\mathrm{dn}\, u. \tag{102}$$

We have also used the formulas (note that they apply to the case $m = \frac{1}{2}$ only):

$$[E(\mathrm{am}\, u|\tfrac{1}{2})]' = \tfrac{1}{2}(\mathrm{cn}^2 u + 1) \qquad [\Pi(\tfrac{x}{x-1};\mathrm{am}\, u|\tfrac{1}{2})]' = \frac{1-x}{1-x\,\mathrm{cn}^2 u}. \tag{103}$$

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The **theta functions** are a collection of four special functions defined via their Fourier expansions; here we will use only one of them, namely

$$\theta_3(z|\tau) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos(2nz), \qquad (104)$$

where τ ($\Im\tau > 0$) is the lattice parameter, once it is chosen, it is usually suppressed. $q = e^{i\pi\tau}$ is the nome. The quasi-periodicity in case of θ_3 is expressed as

$$\theta_3(z + (m + n\tau)\pi) = q^{-n^2}e^{-2inz}\theta_3(z) \qquad (105)$$

for integer n, m (and thus π is an actual, not only quasi, period).

This property allows one to very easily construct meromorphic functions on the elliptic curve (i.e. doubly periodic functions on the complex plane). Indeed, functions of the following types

$$\prod_{i=1}^{n}\frac{\theta_3(z - a_i)}{\theta_3(z - b_i)} \qquad\qquad \sum_{i=1}^{n} R_i\partial_z\ln\theta_3(z - b_i) \qquad (106)$$

are actually doubly periodic (not only quasi) under the following conditions: $\sum a_i - \sum b_i = k\pi$ for integer k, and $\sum R_i = 0$, respectively. The first form is very convenient to use if we have information on the location of zeroes and poles of the elliptic function that we want to construct, while the second form is convenient if the function has only single poles with prescribed residues.

Since θ_3 has a zero at $z = \pi(1 + \tau)/2$, it is convenient to shift the argument in order to have a function which vanishes at $z = 0$. We denote such a function by $\theta(z)$ and its logarithmic derivative by $\phi(z) = \partial\ln\theta(z)$. In addition it is sometimes convenient to also rescale the argument. We use explicitly

$$\theta(z) = \theta_3\left(\frac{\pi z}{k} - \frac{1 + \tau}{2}\pi \middle| \tau\right) \qquad (107)$$

$$\theta(z + k(m + n\tau)) = e^{-i\pi n(\tau n - 1 - \tau + 2z/k)}\theta(z) \qquad (108)$$

$$\phi(z) = \frac{\pi}{k\theta(z)}\theta_3'\left(\frac{\pi z}{k} - \frac{1 + \tau}{2}\pi \middle| \tau\right) \qquad (109)$$

with $k = 2K(\frac{1}{2}), \tau = i$ in the case of $q\bar{q}$ reconstruction and $k = 2\omega$ and $\tau = \omega'/\omega$ in the following appendix. Note that the expressions of the form (106), but with θ instead of θ_3, have poles precisely at all b_i's (and zeroes at a_i's in the former case).

B Other elliptic reconstructions

In this appendix, we will argue how one can reconstruct the GKP folded string from the curve $y^2 = (x^2 - 1)(x^2 - a^2)$ and the generalized $q\bar{q}$ potential from $y^2 = (x^2 - 1)(x - a)(x + 1/a)$ giving justification to the statements made in section 7. We start from giving some very general formulas which we then specialize to the two curves of interest.

B.1 Generalities

Let us briefly review the generic features of reconstructing the classical solution from a general elliptic curve, with the proviso that the Lax matrix is diagonal at $x = \infty$ (so the situation is simpler than for the case of $q\bar{q}$ potential discussed in the main text), and $x = \infty$ is not a branch point of the algebraic curve.

We will denote the (spectral) coordinate on the elliptic curve by u. We will always assume that the passage to the other sheet occurs through the transformation $u \to -u$, i.e.

$$x(-u) = x(u) \qquad y(-u) = -y(u) \tag{110}$$

The branch points will then be located at the half-periods

$$u = 0, \omega, \omega', \omega + \omega' \quad \text{or} \quad u = 0, \frac{1}{2}, \frac{\tau}{2}, \frac{1 + \tau}{2} \tag{111}$$

For solutions completely contained in AdS_3, two of these branch points will correspond to $x = +1, -1$. We will denote these positions by $u = I_+, I_-$. The other points of relevance on the elliptic curve are the images of $x = \infty$: $u = \infty^+$ and $u = \infty^- \equiv -\infty^+$; and the images of $x = 0$: $u = 0^+$ and $u = 0^- \equiv -0^+$.

From the discussion in the main text we know that the lower component of the normalized eigenvector $\Psi_n(w, \bar{w}; u)$ should have a zero at $u = \infty^+$, a pole at $u = \infty^-$ and a further single dynamical pole. Consequently it can be written as

$$\Psi_n(w, \bar{w}; u) = \begin{pmatrix} 1 \\ b(w, \bar{w}) \frac{\theta(u - \infty^+)}{\theta(u - \infty^-)} \cdot \frac{\theta(u + \infty^+ - \infty^- - \gamma)}{\theta(u - \gamma)} \end{pmatrix} \tag{112}$$

where $\gamma \equiv \gamma(w, \bar{w})$ is the position of the single dynamical pole. The Baker-Akhiezer function is again immediate to write:

$$f_{BA}(w, \bar{w}; u) = a(w, \bar{w}) \cdot e^{\phi(u - I_+)w + \phi(u - I_-)\bar{w}} \cdot \frac{\theta(u - \gamma)}{\theta(u - \gamma_0)} \tag{113}$$

where γ_0 is some reference point. The requirement that f_{BA} is doubly periodic in u allows us to determine the position of the dynamical pole, as in (72). We obtain

$$\gamma(w, \bar{w}) = w + \bar{w} + \gamma_0 \tag{114}$$

$a(w, \bar{w})$ and $b(w, \bar{w})$ may be easily reconstructed from the behaviour at $u = \infty^+$ and $u = \infty^-$. The result is

$$\Psi(w, \bar{w}; u) = \begin{pmatrix} \frac{e^{\phi(u-I_+)w+\phi(u-I_-)\bar{w}}}{e^{\phi(\infty^+-I_+)w+\phi(\infty^+-I_-)\bar{w}}} \cdot \frac{\theta(\infty^+-\gamma_0)\theta(u-\gamma)}{\theta(u-\gamma_0)\theta(\infty^+-\gamma)} \\ b\frac{e^{\phi(u-I_+)w+\phi(u-I_-)\bar{w}}}{e^{\phi(\infty^--I_+)w+\phi(\infty^--I_-)\bar{w}}} \cdot \frac{\theta(\infty^--\gamma_0)\theta(u-\infty^+)\theta(u+\infty^+-\infty^--\gamma)}{\theta(u-\gamma_0)\theta(u-\infty^-)\theta(\infty^+-\gamma)} \end{pmatrix} \tag{115}$$

where b is now a constant. Denoting for simplicity the two components by

$$\Psi(w, \bar{w}; u) = \begin{pmatrix} UP(u) \\ b \cdot DN(u) \end{pmatrix} \tag{116}$$

we can put $\hat{\Psi}$ to be equal to[7]

$$\hat{\Psi} = \begin{pmatrix} A_1 \cdot UP(0^+) & A_2 \cdot UP(0^-) \\ A_1 \cdot b \cdot DN(0^+) & A_2 \cdot b \cdot DN(0^-) \end{pmatrix} \tag{117}$$

with $A_{1,2}$ and b arbitrary constants.

We can now obtain explicit expressions for the solution in global AdS_3 spacetime by using (9) and the global AdS_3 formula in (3). We get

$$e^{2it} = \frac{A_2 b}{A_1} \cdot \frac{DN(0^-)}{UP(0^+)} \tag{118}$$

$$e^{2i\psi} = \frac{A_2}{bA_1} \cdot \frac{UP(0^-)}{DN(0^+)} \tag{119}$$

$$\cosh^2 \rho = \frac{A_1 A_2 b \cdot DN(0^-)UP(0^+)}{\det \hat{\Psi}} \tag{120}$$

In the last equation $\det \hat{\Psi}$ is just a pure number.

With these expressions in hand, we will now indicate how the well known solutions – the GKP string and the generalized $q\bar{q}$ potential arise from their algebraic curves. Of course, these solutions are much simpler to obtain directly. For us the main motivation for doing this calculation is to make a clear link with algebraic curves. However, once we would want to obtain solutions of the linear system for the GKP string, we believe that this route is the best (as we failed to directly solve (6) for the GKP folded string).

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[7]This is not the most general expression but will suffice for the examples in the appendices.

B.2 $y^2 = (x^2 - 1)(x^2 - a^2)$ — the GKP folded string

We can uniformize the algebraic curve $y^2 = (x^2 - 1)(x^2 - a^2)$ either using θ functions, as in the main text, or using Weierstrass $\wp$ functions after bringing the curve to the standard Weierstrass form. For completeness we will give explicit formulas here. For the present case we find

$$x(u) = \frac{-1 + 5a^2 + 6(1 - a^2)X(u)}{-5 + a^2 - 6(1 - a^2)X(u)} \tag{121}$$

$$y(u) = \sqrt{\frac{a^2 - 1}{2}} \cdot (1 + x(u)^2) \cdot Y(u) \tag{122}$$

with $X(u) = \wp(u; \{g_2, g_3\})$, $Y(u) = \wp'(u; \{g_2, g_3\})$, where

$$g_2 = \frac{1 + 14a^2 + a^4}{3(-1 + a^2)^2} \qquad g_3 = \frac{1 - 33a^2 - 33a^4 + a^6}{27(-1 + a^2)^3} \tag{123}$$

We find then that $I_- = 0$, $I_+ = \omega$, while $\infty^+ = \omega' - \omega/2$ and $0^+ = -\omega/2$. In order to identify the solution with the folded GKP string it is really enough to just check that $t = const \cdot \tau$, $\psi = const' \cdot \tau$ and $\rho = \rho(\sigma)$.

Let us first consider t and identify the worldsheet dependence following from (118). Apart from the exponent we have the following combination of θ functions:

$$\frac{\theta(0^- + \infty^+ - \infty^- - \gamma)}{\theta(0^+ - \gamma)} \tag{124}$$

Using the explicit locations of these points given above, we find that $0^- + \infty^+ - \infty^- = 0^+ + 2\omega'$ and hence the two θ functions cancel leaving just an additional exponent. Collecting the exponents together we find

$$e^{2it} = \widetilde{const} \cdot e^{const \cdot (\bar{w} - w)} \tag{125}$$

We can get rid of $\widetilde{const}$ through a judicious choice of the constants $A_{1,2}$ and b. This establishes that $t = const \cdot \tau$.

For (119) we get similarly

$$\frac{\theta(0^- - \gamma)}{\theta(0^+ + \infty^+ - \infty^- - \gamma)} \tag{126}$$

Again we find that $0^+ + \infty^+ - \infty^- = 0^- - 2\omega + 2\omega'$ and the same reasoning applies. Consequently we find that

$$e^{2i\psi} = \widetilde{const'} \cdot e^{const' \cdot (\bar{w} - w)} \tag{127}$$

showing that indeed $\psi = const' \cdot \tau$.

Finally, for $\cosh^2 \rho$ we find nontrivial dependence on $w + \bar{w}$ coming both from the θ functions and from the exponential factor. So we get $\rho = \rho(\sigma)$. This is enough to identify the solution with the GKP folded string.

B.3 $y^2 = (x^2 - 1)(x - a)(x + 1/a)$ — the generalized $q\bar{q}$ potential

The algebraic curve can be uniformized similarly as before. We get

$$x(u) = \frac{3 + 4a - 3a^2 - 6(-1 + a^2)X(u)}{3 - 4a - 3a^2 + 6(-1 + a^2)X(u)} \tag{128}$$

$$y(u) = -\sqrt{\frac{a^2 - 1}{2a}} \cdot (1 + x(u))^2 \cdot Y(u) \tag{129}$$

where $X(u) = \wp(u; \{g_2, g_3\})$, $Y(u) = \wp'(u; \{g_2, g_3\})$ but now with

$$g_2 = \frac{(3 + a^2)(1 + 3a^2)}{3(a^2 - 1)^2} \qquad g_3 = \frac{-2a(9a^4 + 14a^2 + 9)}{27(a^2 - 1)^3} \tag{130}$$

We find that $I_- = 0$, $I_+ = \omega + \omega'$, however in the present case ∞^+ is not given in any simple form in terms of the half-periods. However due to the symmetry $x \to -1/x$ of the algebraic curve, which is realized as $u \to \omega + \omega' \pm u$, we can express 0^+ also in terms of ∞^+ ($\infty^- \equiv -\infty^+$ and $0^- \equiv -0^+$ follow immediately):

$$0^+ = \omega + \omega' + \infty^+ \tag{131}$$

Let us now repeat the analysis done for the GKP string. For e^{2it}, our conclusion is unchanged since again

$$0^- + \infty^+ - \infty^- = \infty^+ - \omega - \omega' \equiv 0^+ - 2\omega - 2\omega' \tag{132}$$

so the θ functions cancel. The exponents again lead to $t = const(\bar{w} - w) \propto \tau$.

The situation for $e^{2i\psi}$ is, however, more subtle. We find

$$e^{2i\psi} = \widetilde{const'} \cdot \frac{\theta(0^- - \gamma)}{\theta(0^+ + \infty^+ - \infty^- - \gamma)} \cdot e^{const'(w + \bar{w})} \tag{133}$$

Firstly this is now a function of $w + \bar{w}$ instead of $\bar{w} - w$ as for the GKP string. Secondly, the θ functions no longer cancel and the dependence on σ is quite nontrivial. Thirdly, we find that the requirement of a *real* solution, which corresponds here to requiring that $|e^{2i\psi}| = 1$ for some choice of constants severly restricts the choices of γ_0 and the real form of the worldsheet coordinates (recall that $\gamma = w + \bar{w} + \gamma_0$). Some (nonexhaustive) numerical experimentation leads to the choices that i) $w + \bar{w} = 2i\sigma$ and ii) $\gamma_0 = \infty^+$ or $\gamma_0 = 0^+$. In these cases we get a real $\psi = \psi(\sigma)$.

Let us now proceed to the formula for $\cosh^2 \rho$. Here we find

$$\cosh^2 \rho = const \cdot \frac{\theta^2(0^- + \infty^+ - \infty^- - \gamma)}{\theta^2(0^+ - \gamma)} \cdot e^{-\frac{\pi}{\omega}(w + \bar{w})} \tag{134}$$

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Firstly we see that again this is a function of σ alone. For the case of $\gamma_0 = \infty^+$ the expression turns out to be real and positive. However the θ function in the denominator will have zeroes, which shows that the solution has $\rho \to \infty$ there, which means that it reaches the boundary and hence represents a Wilson loop. Moreover, one can check that the boundary values of ψ at the two edges differ. So the solution with $\gamma_0 = \infty^+$ corresponds exactly to a Wilson loop in global AdS_3, where the boundary lines have some angular separation. This is exactly the case of the generalized $q\bar{q}$ Wilson loop considered in [21, 23, 22] which approximates the ordinary $q\bar{q}$ potential Wilson loop. This identification is consistent with viewing the approximation on the level of algebraic curves as discussed in section 7.

The second choice $\gamma_0 = 0^+$, which does not lead to singularities in the θ functions, unfortunately leads to $\cosh^2 \rho < 0$. Moreover, even complexified, this solution is not periodic so cannot be used as a counterpart of the GKP folded string for the $q\bar{q}$ potential.

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